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## Nonlinear $H_\infty$ state feedback controllers: computation of valid region

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### Abstract

"From a general point of view the state feedback  $H_\infty$  suboptimal control problem is reasonable well-understood. Important problems remain with regard to a priori information of the size of the neighbourhood where the local state feedback  $H_\infty$  problem is solvable, and with regard to the nature of solutions  $V$  to the Hamilton-Jacobi inequalities ... such as properness of  $V$  as a candidate Lyapunov function" Citation van der Schaft, 1992 [vdS92].

The first of these problems is solved regionally (semi-globally) in this paper, and the obtained control laws are implemented in MAPLE.

### 1. Introduction

The local nonlinear state feedback  $H_\infty$  control problem has been solved in the early nineties. Van der Schaft [vdS92] describes the solution process for affine systems using the theory of dissipative systems first introduced by Willems [Wil72]. Isidori [Isi92], and Isidori and Astolfi [IA92b], [IA92a] approach the problem by the theory of differential games. The problem has been solved for more general nonlinear plants by Isidori and Kang [IK95], and Ball, Helton, and Walker [BHW93]. Lukes approximation scheme [Luk69] has been used to compute solutions of the Hamilton-Jacobi inequalities of the local nonlinear state feedback  $H_\infty$  control problem [IK95], [MP95].

From an engineering point of view all these results are insufficient: local control laws are found without any knowledge of boundedness of the closed loop state trajectories, or the size and shape of the neighbourhood where state feedback control works as intended.

State boundedness is an indispensable property of controlled systems for two reasons: state blow-up leads usually to plant or control system failure or damaging, and local control is certainly not applicable when the bound on state trajectories is not known a priori.

The size and shape of the neighbourhood where the implemented control law is meaningful are important design parameters for any practical control purpose. If they do not cover the intended performance envelope of the plant, another control strategy must be chosen.

Finally, any practical oriented control law should allow for inaccurate setting of initial conditions. This kind of initial value robustness may be very important when calculating the secure performance envelope of a closed loop control system.

In the following, given a particular solution  $V$  of the Hamilton-Jacobi inequality of concern, any compact neighbourhood  $\Omega \subset \mathbb{R}^m$  of the equilibrium point which has these beneficial properties is called a **valid region**. We assume without loss of generality that the equilibrium point of concern is at the origin. Moreover, a class of disturbances such that some given  $\Omega$  is a valid region, is called **valid disturbance set**, and it is denoted  $\mathbf{W}^\epsilon$ . A set of initial conditions, denoted  $\Omega^\epsilon \subset \Omega$ , is called **valid initial set** if all state trajectories driven by valid disturbances renders  $\Omega$  a valid region. The problem addressed here is:

**1.1 Problem Formulation** Given a nonlinear state feedback  $H_\infty$  control problem and a formal solution  $V$  to the associated Hamilton-Jacobi inequality, find a valid region  $\Omega$ , a valid disturbance set  $\mathbf{W}^\epsilon$ , and a valid initial set  $\Omega^\epsilon \subset \Omega$  such that every state trajectory  $x(\cdot)$  with initial condition  $x_0 \in \Omega^\epsilon$  subject to disturbances  $w(\cdot) \in \mathbf{W}^\epsilon$  satisfies an  $L_2$  gain less than or equal  $\gamma$ , and approaches the origin as time goes to infinity.

### 2. Local $H_\infty$ state feedback control

Let  $\mathbb{R}^+$  denote the real positive closed time axis  $[0, \infty[$ . In general we consider the plant

$$\dot{x} = X(x, u, w) \quad , \quad z = Z(x, u) \quad (1)$$

where  $x(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^m$  is called the state,  $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^l$  the input,  $w(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^n$  the exogenous input, also called disturbance, and  $z(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$  the performance,

or to be controlled signal. The symbol  $X(x, u, w)$  denotes a smooth vector field on  $\mathbb{R}^m$ , and the vector valued function  $Z(x, u)$  specifies smoothly the performance measure. We assume the equilibrium conditions  $X(0, 0, 0) = 0$  and  $Z(0, 0) = 0$  to hold.

The open loop system  $\dot{x} = X^{\text{open}}(x, w) \equiv X(x, 0, w)$  subject  $w(\cdot) = 0$  is autonomous, and by assumption the origin is an equilibrium point, hence an invariant set. The static feedback control is given by some vector valued function  $a : \mathbb{R}^m \mapsto \mathbb{R}^l$ ,  $u = a(x)$ , where  $a(0) = 0$  is assumed in order to preserve the equilibrium point zero. The closed loop system is given by the equations

$$\dot{x} = X(x, a(x), w) \quad , \quad z = Z(x, a(x)) \quad . \quad (2)$$

Whenever convenient, we use the notation  $x(\cdot)$  for the signal  $x(\cdot, t_0, x_0, u(\cdot), w(\cdot))$ . It is assumed that all signals are  $L_2^{\text{loc}}$ , and that the state exist uniquely for all input, and is a  $C^1$  signal.

Let  $|\cdot|$  denote the usual Euclidean norm on the Banach space  $\mathbb{R}^p$ , then the  $L_2$  norm of any locally square integrable signal  $y(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^p$ ,  $p \in \mathbb{N}$  is for all  $T \in \mathbb{R}^+$  defined

$$\|y\|_T^2 \equiv \int_0^T |y(t)|^2 dt \quad , \quad (3)$$

and the closed loop system (2) has by definition local  $L_2$  gain less than or equal  $\gamma \geq 0$  if

$$\|z\|_T^2 \leq \gamma^2 \|w\|_T^2 + V_a(x_0) \quad (4)$$

for all  $w(\cdot), z(\cdot) \in L_2^{\text{loc}}$ , all  $T \in \mathbb{R}^+$  and all initial conditions  $x_0 \in \mathbb{R}^m$  such that the state trajectories never leave  $\Omega$ . Here the available storage  $V_a : \mathbb{R}^m \mapsto [0, \infty[$  is a non-negative and bounded function with minimum  $V_a(0) = 0$  at the origin [vdS92], [IA92a].

It is known [vdS92] that the local  $L_2$  gain condition is implied by the existence of a non-negative, bounded  $C^1$  storage function  $V : \Omega \mapsto [0, \infty[$  satisfying the closed loop differential dissipation inequality

$$\begin{aligned} \mathbf{H}_\gamma(u, w) &\equiv \frac{d}{dt} V - (\gamma^2 |w|^2 - |z|^2) \\ &= \frac{\partial V}{\partial x} X(x, u, w) \\ &\quad - \gamma^2 |w|^2 + |Z(x, u)|^2 \leq 0 \end{aligned} \quad (5)$$

for all  $t \in \mathbb{R}^+$ , where the Hamiltonian function  $\mathbf{H}_\gamma$  is defined by equation (5). Assuming that  $Z(x, u)$  is such that  $\frac{\partial Z}{\partial u}(0, 0)$  has rank  $l$ , it is known [IK95] that  $\mathbf{H}_\gamma$  has a unique saddle point  $(u_{\min}, w_{\max})$  for all  $x$  and all  $\frac{\partial V}{\partial x}$  near zero, and the extremal functions  $u_{\min}(x, \frac{\partial V}{\partial x})$ ,  $w_{\max}(x, \frac{\partial V}{\partial x})$  are characterized by the equations

$$\begin{aligned} \frac{\partial \mathbf{H}_\gamma}{\partial u}(u_{\min}, w_{\max}) &= 0 \quad , \quad u_{\min}(0, 0) = 0 \\ \frac{\partial \mathbf{H}_\gamma}{\partial w}(u_{\min}, w_{\max}) &= 0 \quad , \quad w_{\max}(0, 0) = 0 \end{aligned} \quad (6)$$

We deduce by the saddle point property (6) that  $u_{\min}$  and  $w_{\max}$  are the best possible state feedback law and the worst possible disturbance respectively.

Hence, we seek for a sufficient small  $\gamma \geq 0$ , and a  $C^1$  storage function  $V$  defined on a sufficient big neighbourhood  $\Omega$  around the origin satisfying the Hamilton-Jacobi inequality [IK95]

$$\begin{aligned} \min_{\gamma} \mathbf{H}_\gamma^{\max}(x, \frac{\partial V}{\partial x}) &= \frac{\partial V}{\partial x} X(x, u_{\min}(x, \frac{\partial V}{\partial x}), w_{\max}(x, \frac{\partial V}{\partial x})) \\ &\quad - \gamma^2 |w_{\max}(x, \frac{\partial V}{\partial x})|^2 + |Z(x, u_{\min}(x, \frac{\partial V}{\partial x}))|^2 \\ &\leq 0 \quad \text{for all } x \in \Omega \end{aligned} \quad (7)$$

This problem can be solved locally by an polynomial expansion algorithm due to Lukes [Luk69].

In the case of input affine systems

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_w(x)w \\ z &= C(x) + D(x)u \end{aligned} \quad (8)$$

(here we have  $A(0) = 0$  and  $C(0) = 0$ ) satisfying  $D^T D > 0$  for all  $x$ , the worst case disturbance and the minimizing input can be found completing the squares, they are for all  $x$  and  $\frac{\partial V}{\partial x}$  given by the expressions [Isi92]

$$\begin{aligned} u_{\min} &= -(D^T D)^{-1} (\frac{1}{2} B_u^T \frac{\partial V}{\partial x} + D^T C) \quad , \\ w_{\max} &= \frac{1}{2\gamma^2} B_w^T \frac{\partial V}{\partial x} \end{aligned} \quad (9)$$

Here we seek for a sufficient small  $\gamma \geq 0$ , and a  $C^1$  storage function  $V$  defined on a sufficient big neighbourhood  $\Omega$  around the origin satisfying the Hamilton-Jacobi inequality [Isi92]

$$\begin{aligned} \min_{\gamma} \mathbf{H}_\gamma^{\max}(\frac{\partial V}{\partial x}, x) &= \frac{\partial V}{\partial x} Q(x) \frac{\partial V}{\partial x}^T + \frac{\partial V}{\partial x} L(x) + K(x) \\ &\leq 0 \quad \text{for all } x \in \Omega \end{aligned} \quad (10)$$

where the quadratic term  $Q(x)$ , the linear term  $L(x)$ , and the constant term  $K(x)$  are defined

$$\begin{aligned} Q(x) &\equiv \frac{1}{4\gamma^2} B_w B_w^T - \frac{1}{4} B_u (D^T D)^{-1} B_u^T \\ L(x) &\equiv A - B_u (D^T D)^{-1} D^T C \\ K(x) &\equiv C^T (I - D (D^T D)^{-1} D^T) C \end{aligned} \quad (11)$$

The existence of a  $C^1$  storage function satisfying (7) or (10) locally guarantees that the input-output map of the closed loop system has  $L_2$  gain less than or equal  $\gamma$  as defined in equation (4) *if and only if every closed loop trajectory is bounded inside  $\Omega$* . Unfortunately, we do not have any priory estimates on the boundedness or on the asymptotic behaviour of the state.

We will impose in the following a detectability assumption on the system: The control system (1) is **zero-detectable** if all bounded trajectories  $x(\cdot)$  subject  $w(\cdot) = 0$  generating the zero-output  $z(\cdot) = 0$  are approaching the origin as  $t \rightarrow \infty$ .

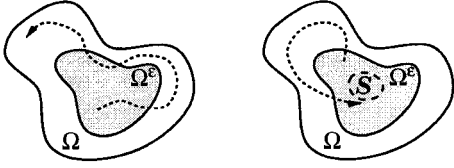


Figure 1: Boundedness of state, asymptotic stability

### 3. Regional $H_\infty$ control

A set  $M \subset \mathbb{R}^m$  is **positive invariant** with respect to the autonomous system (2) if all trajectories starting in  $M$  are defined in the future and never leave  $M$  as time increases. It is **invariant** if  $x(\cdot)$  defined in future and past, and evolves entirely in  $M$ .

It is our purpose to use a formal solution to the Hamilton-Jacobi inequality as a Lyapunov function in order to establish regional stability properties of the  $H_\infty$  state feedback problem. Our new theorem is inspired on the proof of the well known La Salle invariance principle for autonomous systems [SL61], which is connecting the existence of a  $C^1$  Lyapunov function  $V : \mathbb{R}^m \mapsto \mathbb{R}$  with bounded and connected pre-image  $\Omega \equiv V^{-1}([-\infty, c])$  satisfying  $\frac{\partial V}{\partial t} V \leq 0$ , with the positive invariance of  $\Omega$  and the asymptotic stability property of largest invariant set contained in the set where  $\frac{\partial V}{\partial t} V = 0$ .

In order to prove the boundedness of state trajectories we have to restrict ourselves to the class of disturbances

$$\mathbf{W}^\epsilon \equiv \left\{ w(\cdot) \in L_2(\mathbb{R}^+) \mid \|w\|_2^2 \leq \epsilon \right\} . \quad (12)$$

Given some solution  $V$  to the standard  $H_\infty$  Hamilton-Jacobi inequality, the following new lemma will help us to construct the valid sets (left part of figure 1). The lemma is proved in [CMPP96].

**3.1 Lemma** Given a formal  $C^1$  solution  $V$  of the Hamilton-Jacobi inequality (7) or (10), assume that some component of  $V^{-1}([-\infty, c_\Omega])$ ,  $c_\Omega \in \mathbb{R}$ , denoted  $\Omega$ , is connected and bounded.

Then  $\Omega$  is compact and closed loop positive invariant by use of the state feedback law  $a(x) = u_{\min}(x)$  subject to the condition  $w(\cdot) = 0$ .

Pick some  $\epsilon < c_\Omega$ , then the appropriate subset  $\Omega^\epsilon \subset \Omega$  of  $V^{-1}([-\infty, c_\Omega - \epsilon])$  is such that any closed loop trajectory  $x(\cdot)$  with initial condition  $x_0 \in \Omega^\epsilon$  is bounded inside  $\Omega$  if driven by the state feedback law  $a(x) = u_{\min}(x)$ , and by any disturbance  $w(\cdot) \in \mathbf{W}^\epsilon$ .

Following the principal idea of the paper [IK95] as outlined in section 2., we conclude that any  $C^1$  function  $V$  satisfying the Hamilton-Jacobi inequality (7) will also satisfy the  $L_2$  gain (4) in case that the state is bounded inside  $\Omega$ . We take advantage of lemma 3.1 to state the following theorem, and to follow the above explained ideas to proof it. The proof is found in appendix A.

**3.2 Theorem** Assume that some  $C^1$  solution  $V : \Omega \mapsto \mathbb{R}$  of the Hamilton-Jacobi inequality (7) is defined on a bounded and connected component  $\Omega$  of  $V^{-1}([-\infty, c_\Omega])$ ,  $c_\Omega \in \mathbb{R}$ . Assume furthermore that  $\frac{\partial V}{\partial u}(0, 0)$  has rank  $l$ .

Then all closed loop trajectories  $x(\cdot)$  subject  $a(x) = u_{\min}(x)$  with initial condition  $x_0 \in \Omega^\epsilon$  do not leave  $\Omega$  if driven by some  $w(\cdot) \in \mathbf{W}^\epsilon$ , and the system has  $L_2$  gain less than or equal  $\gamma$ .

Moreover, all such  $x(\cdot)$  generated by  $w(\cdot) \in \mathbf{W}^\epsilon$  which are identically zero for all times  $t > T$ ,  $T \in \mathbb{R}$ , approach the biggest closed loop invariant set  $M$  contained in the null set

$$\mathbf{N} \equiv \left\{ x \in \Omega \mid H_\gamma^{\min}(x, \frac{\partial V}{\partial x}) = 0 \right\} .$$

Assume furthermore that the control system (1) is zero-detectable, then  $x(\cdot)$  approaches the origin as  $t \rightarrow \infty$ .

In other words, given a formal solution  $V$  to the associated Hamilton-Jacobi inequality, the valid region  $\Omega$  is given by some bounded (and connected) component of the pre-image  $V^{-1}([-\infty, c])$ , the valid disturbance set  $\mathbf{W}^\epsilon$  is the set of signals which  $L_2$  norm is bounded by  $\epsilon$ , and the valid initial set  $\Omega^\epsilon$  is then given by the appropriate component of the pre-image  $V^{-1}([-\infty, c - \epsilon])$ .

### 4. Lukes approximation

Local solutions of the Hamilton-Jacobi inequality can be obtained by use of an approximation scheme originally developed by Lukes [Luk69] for quadratic cost functions [IK95]. For computational ease, we discuss in the following sections only affine systems although Lukes approximation method can be used for general nonlinear systems. We use the MAPLE implementation [MP95].

Consider the perturbed Hamilton-Jacobi equation

$$\begin{aligned} H_\gamma^{\min}(\frac{\partial V}{\partial x}, x) &= \frac{\partial V}{\partial x} Q(x) \frac{\partial V}{\partial x}^T + \frac{\partial V}{\partial x} L(x) + K(x) \\ &= -\Phi(x) \text{ for all } x \in \Omega , \end{aligned} \quad (13)$$

where  $\Phi(x) : \Omega \mapsto \mathbb{R}^+$  is some positive definite perturbation function which we will use as design parameter to style the size and shape of the valid region.

It is easily seen by (8) (11) that  $L(x)$  is of least first order, and  $K(x)$ ,  $\Phi(x)$  and  $V(x)$  of least second. We make the following analytic data expansions  $V(x) = x^T V^{[2]} x + \sum_{k=3}^{\infty} V^{[k]}(x)$ ,  $Q(x) = Q^{[0]} + \sum_{k=1}^{\infty} Q^{[k]}(x)$ ,  $L(x) = L^{[1]} x + \sum_{k=2}^{\infty} L^{[k]}(x)$ ,  $K(x) = x^T K^{[2]} x + \sum_{k=3}^{\infty} K^{[k]}(x)$ ,  $\Phi(x) = x^T \Phi^{[2]} x + \sum_{k=3}^{\infty} \Phi^{[k]}(x)$  where  $(\cdot)^{[k]}$  denotes  $k$ -th order monomials. Then the perturbed Hamilton Jacoby equality (13) can be rewritten

$$\begin{aligned} 0 &= \sum_{m=2}^{\infty} \left[ \sum_{k=1}^{m-1} \frac{\partial V}{\partial x} [^{m-k+1}] L^{[k]} + K^{[m]} + \Phi^{[m]} \right. \\ &\quad \left. + \left( \sum_{k=0}^{m-2} \sum_{l=1}^{m-k-1} \frac{\partial V}{\partial x} [^{m-k-l+1}] Q^{[k]} \frac{\partial V}{\partial x} [^{l+1}] \right) \right] . \end{aligned} \quad (14)$$

Note that  $\frac{\partial V^{[m]}}{\partial x} \equiv \frac{\partial V^{[m]}}{\partial x}$  are of order  $m - 1$ .

Isolating the second order terms we find the usual Riccati equation of the linearized and perturbed  $H_\infty$  control problem

$$0 = L^{[1]}V^{[2]} + V^{[2]}L^{[1]T} + V^{[2]}Q^{[0]}V^{[2]} + K^{[2]} + \Phi^{[2]} \quad (15)$$

In case that  $V^{[2]}$  is a solution of the Riccati equation (15), the second order terms of the perturbed Hamilton-Jacobi equation (14) vanish, and the  $m$  order terms can be rearranged

$$\begin{aligned} -\frac{\partial V^{[m]}}{\partial x} (L^{[1]} + Q^{[0]}V^{[2]})x &= \sum_{k=2}^{m-1} \frac{\partial V^{[m-k+1]}}{\partial x} L^{[k]} + K^{[m]} + \Phi^{[m]} \\ &+ \sum_{k=1}^{m-2} \sum_{l=1}^{m-k-1} \left( \frac{\partial V^{[m-k-l+1]}}{\partial x} Q^{[k]} \frac{\partial V^{[l+1]}}{\partial x} \right) \quad (16) \\ &\text{for } m = 3, 4, 5, \dots \end{aligned}$$

Assuming that the linearized problem (15) has a stabilizing solution  $V^{[2]}$ , the matrix  $F = L^{[1]} + Q^{[0]}V^{[2]}$  is positive definite and has therefore an inverse. Consequently, the equations (16) can be solved recursively.

## 5. Example

The theory developed in the former sections is now illustrated by a simplified two-dimensional example arising in the field of robotics. We consider the plant

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \sin(x_1) - 2x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \\ z &= \begin{bmatrix} x_2 + x_2^3 \\ x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \end{aligned}$$

with equilibrium point zero.

Note that the linearized system is stabilizable by linear feedback, and we want to compare the quality of a linear controller with a third order controller. The optimal feedback  $L_2$  gain is near 1.2, but we use the suboptimal value  $\gamma = 2$ .

Solving the unperturbed Riccati equation (15) (that is  $\Phi = 0$ ), we find the formal second order storage function

$$V_{\text{lin}}(x) = 0.710x_1^2 + 3.097x_1x_2 + 4.037x_2^2$$

which gives the linear feedback

$$u_{\text{lin}}(x) = -1.421x_1 - 3.097x_2$$

Unfortunately, this storage function satisfies not the Hamilton-Jacobi inequality (7) on a sufficient large area: in figure 2 the set

$$\mathbf{A} \equiv \left\{ x \mid H_{\gamma}^{\min}(x, \frac{\partial V}{\partial x}) < 0 \right\}$$

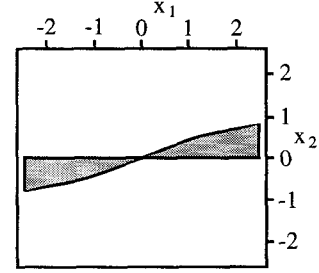


Figure 2: Unperturbed solution

is depicted grey.

Now perturbing the Riccati equation (15) with  $\Phi(x) = 2x_1^2 + x_2^2$  we calculate the formal second order storage function

$$V_{\text{lin,per}}(x) = 0.818x_1^2 + 3.277x_1x_2 + 4.470x_2^2$$

which gives the linear feedback

$$u_{\text{lin,per}}(x) = -1.636x_1 - 3.277x_2$$

Now (7) is satisfied on a sufficiently large area around zero. The set  $\mathbf{A}$  is displayed in figure 3 (left) (white). The pre-images  $\Omega = V^{-1}([0, 2])$  (grey) and  $\Omega^\epsilon = V^{-1}([0, 1])$  with  $\epsilon = 1$  (dark grey) are both inside the set  $\mathbf{A}$ .

Finally, we find a fourth order Lukes approximation for the Hamilton-Jacobi equation (14), using the same perturbation function as in the linear feedback case. We compute the formal storage function

$$\begin{aligned} V_{4\text{th,per}}(x) &= 0.818x_1^2 + 3.277x_1x_2 + 4.470x_2^2 \\ &+ 0.128x_1^2x_2^2 + 0.3113x_1x_2^3 + 0.025x_1^3x_2 \\ &+ 0.002x_1^4 + 0.4124x_2^4 \end{aligned}$$

which gives the third order feedback

$$\begin{aligned} u_{3\text{rd,per}}(x) &= -1.636x_1 - 3.277x_2 - 0.076x_1^2x_2 \\ &- 0.256x_1x_2^2 - 0.008x_1^3 - 0.311x_2^3 \end{aligned}$$

Figure 3 (right) shows a slightly improved set  $\mathbf{A}$  (white). Note that the valid region  $\Omega = V^{-1}([0, 4])$  (grey) is enlarged considerably. The set of valid initial conditions  $\Omega^\epsilon = V^{-1}([0, 1])$  (dark grey) is approximatively the same as for the linear design. Note also that we have chosen  $\epsilon = 3$ , thereby allowing for valid disturbances of three times larger energy than in the linear case.

## 6. Conclusion

In this paper it is shown that state feedback problems involving the regional stabilization of the origin can successfully be recast as generalized formulations of nonlinear local state feedback  $H_\infty$  control problems. Given a formal solution  $V$  to a certain Hamilton-Jacobi inequality, the generalized problem is solved regionally provided  $V$

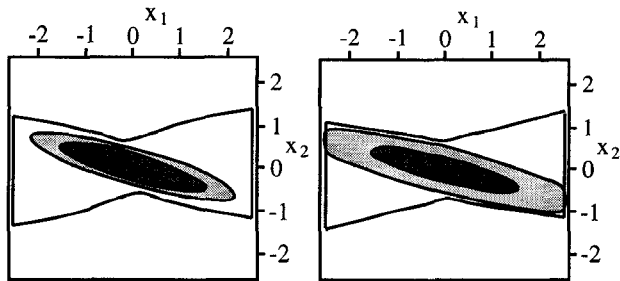


Figure 3: Linear and third order control

is such that some connected component of the pre-image  $V^{-1}(-\infty, c)$  for some  $c \in \mathbb{R}$  is bounded and includes the origin. The plant is assumed to have the standard zero-detectability assumption. Sets of allowable initial conditions and disturbance classes are specified.

Performance is guaranteed in a range of operational conditions, in contrast to local  $H_\infty$  control. Numerical and symbolic computation methods which apply to local  $H_\infty$  theory can without problem be applied in a regional context.

Lukes approximation scheme is explained and implemented in the symbolic language MAPLE, and a formal, local storage function is computed. An example shows that the theory developed in this paper can be used to estimate the valid performance region (performance envelope) of linear as well as nonlinear state feedback controllers.

## A Proof of theorem 3..2

By lemma 3.1 all  $x(\cdot)$  are bounded inside  $\Omega$ . Therefore, the  $L_2$  gain (4) is satisfied.

We show now that all such  $x(\cdot)$  generated by  $w(\cdot) \in \mathbf{W}^\epsilon$  which are identically zero for all times  $t > T$ ,  $T \in \mathbb{R}$ , approach the biggest closed loop invariant set  $M$  contained in the null set  $N$ . By state boundedness, and time invariance of the system, we can assume without loss of generality that  $w(\cdot) = 0$  for all  $t \in \mathbb{R}^+$ . The saddle point property (6) implies that the  $C^1$  solution  $V$  serves as a Lyapunov function for the closed loop dynamics. We have

$$\mathbf{H}_\gamma(u_{\min}, w) \leq \mathbf{H}_\gamma(u_{\min}, w_{\min}) = H_\gamma^{\min} \leq 0 \quad (17)$$

for all  $w(\cdot) \in \mathbf{W}^\epsilon$ . Choosing  $w(\cdot) = 0$  then gives with (5)

$$\frac{d}{dt} V - \gamma^2 |0|^2 + |z|^2 \leq 0 \quad (18)$$

for all such  $x(\cdot)$ . Hence we have  $\frac{d}{dt} V < 0$  for all  $x(\cdot)$  evolving on  $\Omega/N$ . Trajectories on  $N$  are satisfying  $\frac{d}{dt} V = 0$  if and only if  $|z|^2 = |Z(x, u_{\min}(x))|^2 = 0$ , and  $\frac{d}{dt} V < 0$  else.

Now observe that  $V(x)$  by assumption is continuous and defined on a bounded set, hence  $V(x)$  is bounded from below. Given some particular state  $x(\cdot)$ , the storage function  $V(\cdot)$  is decreasing and bounded from below, hence approaches some minimal value, say  $c_\Gamma \in \mathbb{R}$ , as  $t \rightarrow \infty$ . By continuity we conclude that  $V(x) = c_\Gamma$  on the positive limit set  $\Gamma^+$ , and consequently  $\frac{d}{dt} V = 0$  on  $\Gamma^+$ . Rearranging (17) and (18) then

shows that

$$0 \leq |z|^2 \leq H_\gamma^{\min} \leq 0, \quad (19)$$

therefore we conclude that  $\Gamma^+$  is a non-empty subset of the null set  $N$ . But  $\Gamma^+$  is an invariant set, hence contained in the maximal invariant set  $M$ , and consequently any trajectory  $x(\cdot)$  is approaching  $M$  as  $t \rightarrow \infty$ .

We show finally that zero-detectability implies that  $x(\cdot)$  approaches the origin as  $t \rightarrow \infty$ . Clearly any trajectory evolving entirely on  $\Gamma^+$  satisfies by (19) that  $z(\cdot) = 0$ , hence by zero-detectability the origin is approached. Finally, any trajectory with the same limit set  $\Gamma^+ = \{0\}$  is by continuity of the closed loop dynamics forced to approach zero as  $t \rightarrow \infty$ .

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